Recursive Asset Pricing via Wiener-Hopf Methods with an Application to Structural Credit Risk

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Abstract

This paper studies recursive asset-pricing equations that arise in structural credit–risk models. We show that these pricing recursions can be reformulated as Wiener–Hopf type integral equations and analyzed using Fourier transform techniques. Using the model of Ikeda and Igarashi (2016) as an illustration, the bond and equity pricing equations are expressed in this form, which allows the solution to be characterized analytically through the Fourier transform rather than computed by iterative numerical procedures. The uniqueness of the solution is determined by the index of the associated Wiener–Hopf operator. In particular, when the index is zero—as in the model considered here—the solution is uniquely determined. Since this condition is weaker than Blackwell's classical sufficient condition, the approach provides a convenient tool for studying uniqueness in recursive asset–pricing models.

Keywords: Credit risk, Structural model, Option pricing, Wiener–Hopf integral equation, Fourier transform, Blackwell's sufficient condition.

1 Introduction

Recursive asset pricing equations frequently arise in models of corporate securities and credit risk. In many such models, the values of debt and equity are characterized as solutions to recursive functional equations whose computation typically relies on numerical iteration.

This paper proposes a different approach to solving such recursive pricing problems. We show that these equations can be reformulated as Wiener–Hopf type integral equations and analyzed using Fourier transform techniques. This formulation allows the solution to be characterized analytically rather than computed through iterative numerical procedures.

To illustrate the method, we consider the model of Ikeda and Igarashi (2016), which builds on the structural approach to credit risk initiated by Merton (1974). In their framework, a firm issues a zero-coupon bond and redeems the face value at discrete dates by issuing a new bond with the same face value and raising equity if necessary. When repayment is infeasible, creditors may choose either to liquidate the firm or to extend the maturity of the debt.

While other infinite-horizon structural models, such as Black and Cox (1976), Leland (1994), and Leland and Toft (1996), typically assume continuous coupon or face-value payments, Ikeda and Igarashi (2016) consider redemption at discrete dates in order to capture the possibility of repayment extensions. In their model, bond and equity prices are characterized by a recursive functional equation and computed numerically through iteration.

The present paper shows that this pricing recursion can instead be expressed as a Wiener–Hopf type integral equation. The solution can then be obtained analytically using Fourier transform techniques. Moreover, the uniqueness of the solution is determined by the winding number of the associated Wiener–Hopf operator. In particular, when the index is zero—as in the model considered here—the solution is uniquely determined.

Whereas uniqueness for similar equations has often been established using Blackwell's sufficient condition, the present method provides a weaker and more flexible condition.

The remainder of the paper is organized as follows. Section 2 introduces the model and derives the Wiener–Hopf type integral equation. Section 3 presents the solution method, and Section 4 concludes.

2 Model

We follow the framework of Ikeda and Igarashi (2016). The firm's asset value A_t is assumed to follow a geometric Brownian motion under the risk–neutral measure:

$$dA_t = A_t(r\,dt + \sigma\,dW_t), \qquad t \geq 0.$$

At time 0, the firm issues one unit of a bond with maturity ΔT and face value f. At each time $t_i = i\Delta T$ ($i \in \mathbb{N}$), the firm fully redeems the face value only if the asset value A_{t_i} exceeds a benchmark value $\underline{A}$ that is independent of time and the asset path. Redemption is executed by issuing a new bond with the same maturity ΔT and face value f, and by raising equity to cover any shortfall. Although $\underline{A}$ is endogenously determined depending on the case considered below, its explicit determination is not essential for the purposes of this paper and is therefore treated as given.

2.1 Case 1: Immediate Liquidation at Maturity

In this baseline case, creditors do not have the option to offer a repayment extension when the asset value falls below the benchmark $\underline{A}$. If the asset value $\tilde{A}$ is below $\underline{A}$ at maturity, the firm is liquidated and, after deducting costs, creditors receive $(1-\alpha)\tilde{A}$. Under this assumption, let A denote the current asset value, and the bond and equity values immediately after issuance are:

$$F(A;\bar{A}) = e^{-r\Delta T}\,\mathbb{E}\Big[f\,\mathbf{1}_{\{\tilde{A}\geq\bar{A}\}} + \alpha\tilde{A}\,\mathbf{1}_{\{\tilde{A}<\bar{A}\}}\Big], \tag{1}$$

$$S(A;\bar{A}) = e^{-r\Delta T}\,\mathbb{E}\Big[\Big(S(\tilde{A};\bar{A}) - (f - F(\tilde{A};\bar{A}))\Big)\,\mathbf{1}_{\{\tilde{A}\geq\bar{A}\}}\Big]. \tag{2}$$

Since A_t follows geometric Brownian motion,

$$\log(\tilde{A}) \sim N\big(\log(A) + \big(r - \tfrac{1}{2}\sigma^2\big)\,\Delta T,\ \sigma^2\Delta T\big).$$

Define $H = F + S$. Then

$$H(A;\bar{A}) = e^{-r\Delta T}\,\mathbb{E}\Big[H(\tilde{A};\bar{A})\,\mathbf{1}_{\{\tilde{A}\geq\bar{A}\}} + \alpha\tilde{A}\,\mathbf{1}_{\{\tilde{A}<\bar{A}\}}\Big]. \tag{3}$$

Introduce the change of variables:

$$A = \bar{A}e^x, \qquad \tilde{A} = \bar{A}e^y,$$

so that $x = \log(A/\bar{A})$, $y = \log(\tilde{A}/\bar{A})$. Then

$$H(\bar{A}e^x;\bar{A}) = e^{-r\Delta T}\mathbb{E}\big[H(\bar{A}e^y;\bar{A})\,\mathbf{1}_{\{y>0\}} + \alpha\bar{A}e^y\,\mathbf{1}_{\{y\leq 0\}}\big],$$

with

$$y \sim N\big(x + \big(r - \tfrac{1}{2}\sigma^2\big)\,\Delta T,\ \sigma^2\Delta T\big).$$

Define

$$H(\bar{A}e^x) = \begin{cases}\varphi(x), & x > 0,\\ \varphi_1(x), & x \leq 0,\end{cases} \qquad f(x) = e^{-r\Delta T}\,\mathbb{E}\big[\alpha\bar{A}e^y\,\mathbf{1}_{\{y\leq 0\}}\big].$$

Then

$$\varphi(x) = f(x) + e^{-r\Delta T}\mathbb{E}\big[\varphi(y)\mathbf{1}_{\{y>0\}}\big]$$

for $x > 0$.

Writing out the integral,

$$\varphi(x) = f(x) + e^{-r\Delta T}\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2\Delta T}}\exp\left(-\frac{(y-(x+(r-\frac{1}{2}\sigma^2)\Delta T))^2}{2\sigma^2\Delta T}\right)\varphi(y)\,dy$$

and let

$$k(x) = e^{-r\Delta T}\frac{1}{\sigma\sqrt{\Delta T}\sqrt{2\pi}}\exp\left(-\frac{(x+(r-\frac{1}{2}\sigma^2)\Delta T)^2}{2\sigma^2\Delta T}\right), \tag{4}$$

then we obtain the integral equation:

$$\varphi(x) - \int_0^\infty k(x-y)\varphi(y)\,dy = f(x). \tag{5}$$

For $x \leq 0$,

$$\varphi_1(x) = f(x) + e^{-r\Delta T}\mathbb{E}\big[\varphi(y)\mathbf{1}_{\{y>0\}}\big].$$

Therefore, determining H satisfying (3) is equivalent to solving the integral equation (5) for $\varphi(x)$ on $x > 0$.

2.2 Case 2: Repayment Extension upon Insolvency

This case extends Case 1. At maturity, if A_{t_i} falls below $\underline{A}$, the firm defaults in Case 1. In Case 2, however, the creditor may either liquidate the firm or extend the maturity by an additional ΔT. If the maturity is extended, the next redemption date is postponed by ΔT. If the firm again fails to redeem at that date, the creditor may extend the maturity once more, and this process may repeat indefinitely. In addition, when repayment is postponed, no new bonds or equity are issued.[1]

Under these assumptions, the maturity structure of debt immediately after the bond issuance (or maturity–extension decision date) does not depend on calendar time or on the past repayment history (i.e., the past path of the asset value). Hence the model is again a time–homogeneous infinite–horizon model.

Assume that the lower bound of asset values for full repayment is $\underline{A}$, and that the upper bound below which the creditor chooses to extend the maturity is $\bar{A}$, with $\underline{A} \leq \bar{A}$. Let A denote the asset value at the bond issuance date, and let $\tilde{A}$ denote the asset value after ΔT. Let the bond and stock price be denoted by $F(A)$ and $S(A)$, respectively.[2] Then

$$F(A;\bar{A},\underline{A}) = e^{-r\Delta T}\mathbb{E}\Big[P(\tilde{A})\,\mathbf{1}_{\{\tilde{A}\geq\bar{A}\}} + F(\tilde{A})\,\mathbf{1}_{\{\tilde{A}<\bar{A}\}}\Big], \tag{6}$$

where

$$P(A) = \begin{cases} f, & A > \underline{A}, \\ \alpha A, & \bar{A} \leq A \leq \underline{A}, \end{cases}$$

and

$$S(A;\bar{A},\underline{A}) = e^{-r\Delta T}\mathbb{E}\Big[(S(\tilde{A}) - (f - F(\tilde{A})))\mathbf{1}_{\{\tilde{A}\geq\underline{A}\}} + S(\tilde{A})\,\mathbf{1}_{\{\tilde{A}<\bar{A}\}}\Big].$$

As a special limiting case, consider $\alpha = 0$. In this case, liquidation yields nothing to the creditor, so extending the maturity is always weakly optimal whenever the firm cannot redeem.

[1] In the following, the phrase "immediately after the bond issuance date" also includes cases in which bond issuance does not occur because repayment is postponed.

[2] Functions used for pricing in this subsection may share notation with those in the previous subsection, but they represent different functions.

Default therefore never occurs, and the total value of debt and equity equals the asset value. The repayment condition

$$S(\tilde{A}) - (f - F(\tilde{A})) > 0$$

then reduces to $\tilde{A} > f$, so the endogenous threshold becomes f. Hence,

$$P(A) = f \qquad \text{for } A \geq \bar{A},$$

and

$$F(A; f, f) = e^{-r\Delta T}\mathbb{E}\Big[f\,\mathbf{1}_{\{\tilde{A}\geq f\}} + F(\tilde{A})\,\mathbf{1}_{\{\tilde{A}<f\}}\Big],$$
$$S(A; f, f) = e^{-r\Delta T}\mathbb{E}\Big[(\tilde{A} - f)\,\mathbf{1}_{\{\tilde{A}\geq f\}} + S(\tilde{A})\,\mathbf{1}_{\{\tilde{A}<f\}}\Big].$$

We now return to the case of a general α and focus on the bond pricing equation(6). Introduce the change of variables[3]:

$$A = \bar{A}e^{-x} \quad\Longleftrightarrow\quad x = \log\left(\frac{\bar{A}}{A}\right), \qquad \tilde{A} = \bar{A}e^{-y} \quad\Longleftrightarrow\quad y = \log\left(\frac{\bar{A}}{\tilde{A}}\right).$$

Then equation (6) becomes

$$F(\bar{A}e^{-x}) = e^{-r\Delta T}\mathbb{E}\big[P(\bar{A}e^{-y})\,\mathbf{1}_{\{y\leq 0\}} + F(\bar{A}e^{-y})\,\mathbf{1}_{\{y>0\}}\big],$$

where

$$y \sim N\left(x - \left(r - \frac{1}{2}\sigma^2\right)\Delta T,\ \sigma^2\Delta T\right).$$

Define

$$F(\bar{A}e^{-x}) = \begin{cases} \varphi(x), & x > 0, \\ \varphi_1(x), & x \leq 0, \end{cases} \qquad f(x) = e^{-r\Delta T}\mathbb{E}\big[P(\bar{A}e^{-y})\,\mathbf{1}_{\{y\leq 0\}}\big].$$

Then (6) yields

$$\varphi(x) = f(x) + e^{-r\Delta T}\mathbb{E}\big[\varphi(y)\,\mathbf{1}_{\{y>0\}}\big].$$

Writing out the expectation explicitly,

$$\varphi(x) = f(x) + e^{-r\Delta T}\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2\Delta T}}\exp\left(-\frac{\big(y - (x - (r - \frac{1}{2}\sigma^2)\Delta T)\big)^2}{2\sigma^2\Delta T}\right)\varphi(y)\,dy.$$

Define the kernel

$$k(x) = e^{-r\Delta T}\frac{1}{\sigma\sqrt{\Delta T}\sqrt{2\pi}}\exp\left(-\frac{\big(x - (r - \frac{1}{2}\sigma^2)\Delta T\big)^2}{2\sigma^2\Delta T}\right), \tag{7}$$

so that the integral equation becomes

$$\varphi(x) - \int_0^\infty k(x-y)\,\varphi(y)\,dy = f(x). \tag{8}$$

For $x \leq 0$,

$$\varphi_1(x) = f(x) + e^{-r\Delta T}\mathbb{E}\big[\varphi(y)\,\mathbf{1}_{\{y>0\}}\big].$$

Therefore, determining F satisfying (6) is equivalent to solving the integral equation (8) for $\varphi(x)$ on $x > 0$.

[3]Note that this transformation differs from that used in Case 1.

2.3 Wiener–Hopf Equation

The integral equations in (5) and (8) are Wiener–Hopf (WH) equations.[4] In this subsection, we state the key results related to the WH equation that are essential for this paper.

The operator appearing on the left-hand side of (5) or (8),

$$(\mathbf{I}-\mathbf{K})\varphi(x) := \varphi(x) - \int_0^\infty k(x-y)\,\varphi(y)\,dy,$$

is called the *WH operator*. The function

$$1-K(\xi) := 1 - \int_{-\infty}^\infty k(x)e^{-i\xi x}\,dx$$

is called the *symbol* of the WH operator $\mathbf{I}-\mathbf{K}$. The winding number of the symbol,

$$\begin{aligned}\kappa_K &= \frac{1}{2\pi}\int_{-\infty}^\infty d_\xi \arg(1-K(\xi)) \\ &= \frac{1}{2\pi}\left[\arg(1-K(\xi))\right]_{-\infty}^\infty,\end{aligned}$$

is called the *index* of the WH operator.

In what follows, we use the Fourier transform

$$\mathcal{F}k(\xi) = \int_{-\infty}^\infty k(x)e^{-i\xi x}\,dx,$$

and its inverse

$$\mathcal{F}^{-1}K(x) = \frac{1}{2\pi}\int_{-\infty}^\infty K(\xi)e^{i\xi x}\,d\xi.$$

We now state the main result for WH equations.

Theorem 2.1 (Wiener–Hopf Equation)**.** *Consider the WH equation for an unknown function* φ*:*

$$\varphi(x) - \int_0^\infty k(x-y)\,\varphi(y)\,dy = f(x), \qquad (x>0).$$

Assume that the index of its WH operator satisfies $\kappa_K = 0$*. Then the solution is uniquely determined and is given by*

$$\varphi(x) = f(x) + \int_0^\infty l(x,y)\,f(y)\,dy,$$

where the kernel $l(x,y)$ *is defined by*

$$l(x,y) = l_+(x-y) + l_-(x-y) + \int_0^{\min(x,y)} l_+(x-z)\,l_-(z-y)\,dz.$$

The functions $l_\pm(x)$ *are obtained as follows. Let*

$$K(\xi) = \mathcal{F}k(\xi)$$

and

$$g(x) = \mathcal{F}^{-1}\big(\log(1-K(\xi))\big)(x).$$

[4] Historically, this type of equation originated in the study of light propagation in cosmic space and was generalized and analysed by Wiener and Hopf in 1931.

Then

$$K_\pm(\xi) = 1 - \exp\left(\pm \int_0^{\pm\infty} g(x)\, e^{-i\xi x}\, dx\right),$$

and

$$l_\pm(x) = \mathcal{F}^{-1}\left(\frac{K_\pm(\xi)}{1 - K_\pm(\xi)}\right)(x),$$

where $l_+(x) = 0$ *for* $x < 0$ *and* $l_-(x) = 0$ *for* $x > 0$. □

The proof is given in Section 3 following the method of Krein (1960).

Let

$$f_0(x) = \begin{cases} f(x), & x > 0, \\ 0, & x \le 0, \end{cases} \qquad F_0(\xi) = \mathcal{F} f_0(\xi).$$

Then from Theorem 2.1, we obtain the following.

Corollary 2.2. *If the index of the WH operator is zero, then*

$$\varphi_0(x) = \mathcal{F}^{-1}\left(\frac{F_0(\xi)}{1 - K(\xi)}\right)(x)$$

satisfies

$$\varphi_0(x) = \begin{cases} \varphi(x), & x > 0, \\ 0, & x \le 0. \end{cases}$$

That is, the solution $\varphi(x)$ *of the WH equation is uniquely determined on the positive half-line by the inverse Fourier transform of* $\Phi_0(\xi) = F_0(\xi)/(1 - K(\xi))$.

□

To apply Theorem 2.1 to our model, we verify the index condition.

Corollary 2.3. *The WH operators generated by the kernels defined in* (4) *and* (7) *both have index zero.* □

The proofs are also given in Section 3.

3 Solution Method for Wiener–Hopf Type Integral Equations

3.1 Several Types of Integral Equations

We compare four types of integral equations. Let $k, f \in L^1(\mathbb{R})$[5] be given functions, with φ be the unknown function. Consider the following equations:

$$\varphi(x) - \int_{-\infty}^{\infty} k(x-y)\varphi(y)\, dy = f(x), \qquad -\infty < x < \infty, \tag{9}$$

$$\varphi(x) - \int_0^{x} k(x-y)\varphi(y)\, dy = f(x), \qquad x > 0, \tag{10}$$

$$\varphi(x) - \int_x^{\infty} k(x-y)\varphi(y)\, dy = f(x), \qquad x > 0, \tag{11}$$

$$\varphi(x) - \int_0^{\infty} k(x-y)\varphi(y)\, dy = f(x), \qquad x > 0. \tag{12}$$

[5] We denote by $L^1(I)$ the set of all functions f on an interval I such that $\int_I |f(x)|\, dx < \infty$ in the sense of Lebesgue integration. We write $\mathbb{R}$ for the set of all real numbers and $\mathbb{C}$ for the set of all complex numbers, and define

$$\mathbb{C}_+ = \{\, \xi \in \mathbb{C} \mid \operatorname{Im} \xi > 0 \,\}, \qquad \mathbb{C}_- = \{\, \xi \in \mathbb{C} \mid \operatorname{Im} \xi < 0 \,\}.$$

Although these equations appear similar, their integration ranges differ and lead to different analytical treatments.

Equation (9) has the full real line as its integration interval. Applying the Fourier transform, and denoting the transforms of k, φ, and f by $K(\xi)$, $\Phi(\xi)$, and $F(\xi)$, respectively, the convolution theorem yields $(1-K(\xi))\Phi(\xi) = F(\xi)(\xi \in \mathbb{R})$. If $1-K(\xi) \neq 0 \quad (\forall\, \xi \in \mathbb{R})$, then dividing both sides by $1-K(\xi)$ and applying the inverse Fourier transform gives a unique solution $\varphi \in L^p(\mathbb{R})$ for $1 \leq p \leq \infty$.[6]

Equations (10) and (11), with integration intervals $[0,x]$ and $[x,\infty)$, can be solved by adapting the technique for (9). Extend f and φ by zero on $x \leq 0$. For equation (10), set $k(x) = 0$ for $x \leq 0$, and for equation (11), set $k(x) = 0$ for $x > 0$.

Under these extensions, equations (10) and (11) can be rewritten in the form of equation (9). For later use, we summarize the uniqueness conditions as follows.

Theorem 3.1. *Let $k \in L^1(\mathbb{R})$, and let $K(\xi)$ denote its Fourier transform. Assume that*

$$1 - K(\xi) \neq 0 \qquad (\xi \in \mathbb{R}).$$

Then the following hold:

(i) Equation (10) *has a unique solution $\varphi \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, for $x > 0$ if and only if*

$$1 - K(\xi) \neq 0 \qquad (\xi \in \mathbb{C}_-).$$

(ii) Equation (11) *has a unique solution $\varphi \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, for $x > 0$ if and only if*

$$1 - K(\xi) \neq 0 \qquad (\xi \in \mathbb{C}_+).$$

□

Proof. See Kamimura (2001) for a proof. □

However, equation (12) cannot be reduced to any of the previous three forms by simple extensions of the functions. In what follows, we decompose the left-hand side of equation (12) into forms related to equations (10) and (11), which will be the key step in proving Theorem 2.1.

3.2 Proof of Theorem 2.1

This subsection follows Kamimura (2001). We first present several theorems needed for the proof.

In general, the product $(\mathbf{I}-\mathbf{K}_1)(\mathbf{I}-\mathbf{K}_2)$ of two WH operators does not yield another WH operator. However, the following lemma provides a useful exception.

Lemma 3.2. *Consider two WH operators*

$$(\mathbf{I}-\mathbf{K}_i)\varphi(x) := \varphi(x) - \int_0^\infty k_i(x-y)\varphi(y)\,dy, \qquad (i = 1,2),$$

and assume that at least one of the following holds:

$$k_1(x) = 0 \quad (x > 0), \qquad k_2(x) = 0 \quad (x < 0).$$

Then the product $(\mathbf{I}-\mathbf{K}_1)(\mathbf{I}-\mathbf{K}_2)$ is also a WH operator, and its symbol is

$$(1 - K_1(\xi))(1 - K_2(\xi)).$$

□

[6] The justification of this method, including the uniqueness of the solution, was established by Wiener (1932).

Proof. Assume $k_1(x) = 0$ for $x > 0$ (the case $k_2(x) = 0$ for $x < 0$ is proved in the same way). Then

$$\begin{aligned}(\mathbf{I}-\mathbf{K}_1)(\mathbf{I}-\mathbf{K}_2)\varphi(x) &= \varphi(x) - \int_0^\infty (k_1(x-y)+k_2(x-y))\varphi(y)\,dy \\ &\quad + \int_0^\infty \varphi(y)\left(\int_0^\infty k_1(x-z)k_2(z-y)\,dz\right)dy.\end{aligned}$$

Since $k_1(x) = 0$ for $x > 0$,

$$\begin{aligned}\int_0^\infty k_1(x-z)k_2(z-y)\,dz &= \int_{-y}^\infty k_1(x-y-w)k_2(w)\,dw \\ &= \int_{-\infty}^\infty k_1(x-y-w)k_2(w)\,dw \\ &= (k_1 * k_2)(x-y).\end{aligned}$$

Let

$$k := k_1 + k_2 - k_1 * k_2.$$

Then

$$\begin{aligned}(\mathbf{I}-\mathbf{K}_1)(\mathbf{I}-\mathbf{K}_2)\varphi(x) &= \varphi(x) - \int_0^\infty (k_1(x-y)+k_2(x-y))\varphi(y)\,dy \\ &\quad + \int_0^\infty (k_1 * k_2)(x-y)\varphi(y)\,dy \\ &= \varphi(x) - \int_0^\infty k(x-y)\varphi(y)\,dy.\end{aligned}$$

Thus the product is again a WH operator. Moreover, by the convolution identity,

$$\begin{aligned}1 - \int_{-\infty}^\infty k(x)e^{-i\xi x}\,dx &= 1 - K_1(\xi) - K_2(\xi) + K_1(\xi)K_2(\xi) \\ &= (1-K_1(\xi))(1-K_2(\xi)).\end{aligned}$$

□

The solution method for WH equations presented below uses Lemma 3.2 to decompose a given WH operator into two special WH operators. The way of decomposing is given by the following lemma.

Lemma 3.3. *Let* $(\mathbf{I}-\mathbf{K})$ *be a WH operator whose symbol satisfies* $1-K(\xi) \neq 0$ *for* $\xi \in \mathbb{R}$*, and assume that its index is* $\kappa_K = 0$*. Then* $1-K(\xi)$ *admits a unique factorization of the form*

$$1-K(\xi) = (1-K_-(\xi))(1-K_+(\xi)), \qquad (\xi \in \mathbb{R}),$$

where each $1-K_\pm(\xi)$ *is analytic in* $\mathbb{C}_\mp$*, continuous on the closure* $\overline{\mathbb{C}}_\mp$*, has no zeros there, and can be represented as*

$$1-K_+(\xi) = 1 - \int_0^\infty k_+(x)e^{-i\xi x}\,dx, \qquad (\xi \in \mathbb{R}),$$

$$1-K_-(\xi) = 1 - \int_{-\infty}^0 k_-(x)e^{-i\xi x}\,dx, \qquad (\xi \in \mathbb{R}),$$

for some $k_\pm \in L^1(0,\pm\infty)$. □

The proof of Lemma 3.3 is essentially based on the following theorems.

Theorem 3.4. *Let D be a domain in the complex plane $\mathbb{C}$ such that $0 \in D$, and let $\phi(z)$ be analytic on D with $\phi(0) = 0$. If $f \in L^1(\mathbb{R})$ is such that the closed curve*

$$z = \mathcal{F}f(\xi), \qquad -\infty \le \xi \le \infty,$$

lies in D, then there exists $g \in L^1(\mathbb{R})$ such that

$$\phi(\mathcal{F}f(\xi)) = \mathcal{F}g(\xi), \qquad (\xi \in \mathbb{R}).$$

□

Theorem 3.5. *In addition to the assumptions of Theorem 3.4, assume that $f(x) = 0$ for $x < 0$ and $\mathcal{F}f(\xi) \in D$ for $\xi \in \overline{\mathbb{C}}_-$. Then the function $g \in L^1(\mathbb{R})$ obtained in Theorem 3.4 satisfies $g(x) = 0$ for $x < 0$.* □

(The proofs of Theorems 3.4 and 3.5 are given in Kamimura (2001).)

Proof of Lemma 3.3. Since $1 - K(\xi) \ne 0$ for $\xi \in \mathbb{R}$ and the WH index is zero, we regard $\phi(z) = \log(1 - z)$ as a single-valued analytic function on the Riemann surface obtained by cutting along $z \ge 1$. Then the curve $K(\xi)$, $-\infty \le \xi \le \infty$, lies on this surface. By Theorem 3.4, there exists $g \in L^1(\mathbb{R})$ such that

$$1 - K(\xi) = \exp\left[\int_{-\infty}^{\infty} g(x)e^{-i\xi x}\,dx\right].$$

Define

$$1 - K_+(\xi) := \exp\left[\int_0^{\infty} g(x)e^{-i\xi x}\,dx\right], \qquad 1 - K_-(\xi) := \exp\left[\int_{-\infty}^{0} g(x)e^{-i\xi x}\,dx\right].$$

These functions are analytic in $\mathbb{C}_\mp$, continuous on $\overline{\mathbb{C}}_\mp$, and have no zeros there.

From

$$K_+(\xi) = 1 - \exp\left[\int_0^{\infty} g(x)e^{-i\xi x}\,dx\right],$$

we apply Theorems 3.4 and 3.5 with $\phi(z) = 1 - \exp(z)$ and $D = \mathbb{C}$, and obtain $k_+ \in L^1(0, \infty)$. By a similar argument, we obtain $k_- \in L^1(-\infty, 0)$.

Proof of Theorem 2.1. We first prove the uniqueness of the solution. Let $(\mathbf{I} - \mathbf{K})$ be a WH operator whose symbol $1-K(\xi)$ is factorized as in Lemma 3.3, and define WH operators $(\mathbf{I}-\mathbf{K}_\pm)$ by

$$(\mathbf{I} - \mathbf{K}_\pm)\varphi(x) := \varphi(x) - \int_0^{\infty} k_\pm(x - y)\varphi(y)\,dy. \tag{9}$$

Then

$$\begin{aligned}(\mathbf{I} - \mathbf{K}_+)\varphi(x) &= \varphi(x) - \int_0^{\infty} k_+(x - y)\varphi(y)\,dy \\ &= \varphi(x) - \int_0^{x} k_+(x - y)\varphi(y)\,dy,\end{aligned}$$

so the integral equation

$$(\mathbf{I} - \mathbf{K}_+)\varphi(x) = f(x)$$

has the same form as (10). Moreover, by Lemma 3.3, its symbol has no zeros, and hence Theorem 3.1 yields uniqueness of the solution. Similarly,

$$\begin{aligned}(\mathbf{I} - \mathbf{K}_-)\varphi(x) &= \varphi(x) - \int_0^{\infty} k_-(x - y)\varphi(y)\,dy \\ &= \varphi(x) - \int_x^{\infty} k_-(x - y)\varphi(y)\,dy,\end{aligned}$$

so the integral equation

$$(\mathbf{I}-\mathbf{K}_-)\varphi(x)=f(x)$$

has the same form as (11), and for the same reason it has a unique solution.

By Lemma 3.2 we have

$$(\mathbf{I}-\mathbf{K})=(\mathbf{I}-\mathbf{K}_-)(\mathbf{I}-\mathbf{K}_+),$$

and therefore, when $\kappa_K=0$, the solution of the integral equation (12) can be written uniquely as

$$\varphi(x)=(\mathbf{I}-\mathbf{K}_+)^{-1}(\mathbf{I}-\mathbf{K}_-)^{-1}f(x).$$

Next we derive an explicit representation of the solution. Applying Theorem 3.5 with $D=\mathbb{C}\setminus\{1\}$ and $\phi(z)=z/(1-z)$, we obtain

$$\frac{1}{1-K_\pm(\xi)}=1+\int_0^{\pm\infty}l_\pm(x)e^{-i\xi x}\,dx,\qquad(\xi\in\mathbb{R}),$$

for some $l_\pm\in L^1(0,\pm\infty)$. Using this, we take the Fourier transform of both sides of

$$(\mathbf{I}-\mathbf{K}_\pm)\varphi(x)=f(x)$$

to obtain

$$(1-K_\pm(\xi))\Phi(\xi)=F(\xi),\qquad(\xi\in\mathbb{R}).$$

Since $1-K_\pm(\xi)\neq0$ for $\xi\in\mathbb{R}$, we divide by $1-K_\pm(\xi)$ to get

$$\begin{aligned}\Phi(\xi)&=F(\xi)\frac{1}{1-K_\pm(\xi)}\\&=F(\xi)\left(1+\int_0^{\pm\infty}l_\pm(x)e^{-i\xi x}\,dx\right).\end{aligned}$$

Taking inverse Fourier transforms, we have

$$\varphi(x)=f(x)+\int_0^\infty l_\pm(x-y)f(y)\,dy.$$

Thus

$$(\mathbf{I}-\mathbf{K}_\pm)^{-1}f(x)=f(x)+\int_0^\infty l_\pm(x-y)f(y)\,dy,\qquad(x\geq0).$$

Therefore,

$$\begin{aligned}\varphi(x)&=(\mathbf{I}-\mathbf{K}_+)^{-1}(\mathbf{I}-\mathbf{K}_-)^{-1}f(x)\\&=f(x)+\int_0^\infty l_-(x-y)f(y)\,dy\\&\quad+\int_0^\infty l_+(x-y)\left\{f(y)+\int_0^\infty l_-(y-z)f(z)\,dz\right\}dy\\&=f(x)+\left[\int_0^\infty l_-(x-y)f(y)\,dy+\int_0^\infty l_+(x-y)f(y)\,dy\right.\\&\qquad\left.+\int_0^\infty\left(\int_0^\infty l_+(x-z)l_-(z-y)\,dz\right)f(y)\,dy\right].\end{aligned}$$

Using $l_\pm(\mp x)=0$ for $x\geq0$, we finally obtain

$$\varphi(x)=f(x)+\int_0^\infty l(x,y)f(y)\,dy,$$

where

$$l(x,y):=l_+(x-y)+l_-(x-y)+\int_0^{\min(x,y)}l_+(x-z)l_-(z-y)\,dz.$$

3.3 Proof of Corollary 2.2

Consider the WH equation

$$(\mathbf{I}-\mathbf{K})\varphi(x)=f(x), \qquad x>0.$$

Using the operators $\mathbf{K}_\pm$ obtained in (9), this can be written as

$$(\mathbf{I}-\mathbf{K}_-)(\mathbf{I}-\mathbf{K}_+)\varphi(x)=f(x).$$

Set

$$(\mathbf{I}-\mathbf{K}_+)\varphi(x)=p(x),$$

so that

$$(\mathbf{I}-\mathbf{K}_-)p(x)=f(x).$$

Define

$$f_0(x):=\begin{cases} f(x), & x>0,\\ 0, & \text{otherwise,}\end{cases}$$

and consider the equation

$$(\mathbf{I}-\mathbf{K}_-)p_0(x)=f_0(x). \tag{10}$$

By taking the Fourier transform of both sides of (10), we obtain

$$(1-K_-(\xi))\,\mathcal{F}p_0(\xi)=\mathcal{F}f_0(\xi),$$

and

$$\mathcal{F}p_0(\xi)=\frac{\mathcal{F}f_0(\xi)}{1-K_-(\xi)}. \tag{11}$$

Therefore,

$$p_0(x)=\mathcal{F}^{-1}\left(\frac{\mathcal{F}f_0(\xi)}{1-K_-(\xi)}\right).$$

Next, we extend the domain of $(\mathbf{I}-\mathbf{K}_+)\varphi(x)=p(x)$ and consider

$$(\mathbf{I}-\mathbf{K}_+)\varphi_0(x)=p_0(x), \tag{12}$$

where

$$\varphi_0(x):=\begin{cases} \varphi(x), & x>0,\\ 0, & \text{otherwise.}\end{cases}$$

Taking the Fourier transform of both sides of (12) and using (11), we obtain

$$\begin{aligned}(1-K_+(\xi))\,\mathcal{F}\varphi_0(\xi)&=\mathcal{F}p_0(\xi)\\ &=\frac{\mathcal{F}f_0(\xi)}{1-K_-(\xi)}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{F}\varphi_0(\xi)&=\frac{\mathcal{F}f_0(\xi)}{(1-K_+(\xi))(1-K_-(\xi))}\\ &=\frac{\mathcal{F}f_0(\xi)}{1-K(\xi)}.\end{aligned}$$

Hence

$$\varphi_0(x)=\mathcal{F}^{-1}\left(\frac{\mathcal{F}f_0(\xi)}{1-K(\xi)}\right),$$

which gives the desired expression.

3.4 Proof of Corollary 2.3

In either case, it suffices to verify that $1 - K(\xi) \neq 0$ for $-\infty < \xi < \infty$ and that the WH index is equal to 0.

Let

$$R = \exp(-r\Delta T) \ (< 1), \qquad b = \sigma\sqrt{\Delta T}, \qquad c = (r - \tfrac{1}{2}\sigma^2)\Delta T.$$

Case 1

We have

$$k(x) = R \cdot \frac{1}{b\sqrt{2\pi}} \exp\left(-\frac{(x+c)^2}{2b^2}\right),$$

so its Fourier transform is

$$K(\xi) = R \exp\left(-\tfrac{b^2}{2}\xi^2 + ic\xi\right).$$

Since $K(\xi) \to 0$ as $\xi \to \pm\infty$, the curve $1 - K(\xi)$ begins and ends at the point 1 in the complex plane.

Moreover, for every real ξ,

$$\Re(1 - K(\xi)) = 1 - Re^{-b^2\xi^2/2} \cos(c\xi) \geq 1 - R > 0.$$

Thus the curve $1 - K(\xi)$ lies entirely in the right half-plane and never crosses the imaginary axis. It therefore does not wind around the origin. Hence the winding number is zero, and $\kappa = 0$.

Case 2

Here

$$k(x) = R \cdot \frac{1}{b\sqrt{2\pi}} \exp\left(-\frac{(x-c)^2}{2b^2}\right),$$

so its Fourier transform is

$$K(\xi) = R \exp\left(-\tfrac{b^2}{2}\xi^2 - ic\xi\right).$$

Compared with the function $K(\xi)$ in Case 1, the function $K(\xi)$ here is its complex conjugate for each real ξ. Hence the curve $1 - K(\xi)$ in Case 2 is the reflection of the curve in Case 1 with respect to the real axis. In particular, it remains in the right half-plane, starts and ends at 1, and does not wind around the origin. Therefore the WH index is again $\kappa = 0$.

3.5 Discussion

Theorem 2.1 and Corollary 2.3 together show that equations (5) and (8) in the original model admit unique solutions. It is worth noting, however, that substituting the model-specific functions into the explicit representation in Theorem 2.1 does not yield closed-form expressions that are analytically tractable.

An important observation made in the course of proving Theorem 2.1 is that any WH operator with index zero can be decomposed into a product of two special WH operators. Because an integral equation associated with such an operator can be solved by applying the Fourier transform, performing a simple algebraic manipulation, and then applying the inverse transform, Corollary 2.2 shows that any integral equation whose WH operator has index zero can be solved in this straightforward manner without explicitly carrying out the decomposition. Although Theorem 2.1 might suggest that one needs to repeatedly decompose and recombine the operator's symbol, such decomposition is not required when the goal is merely to obtain the solution. Numerical integration is still needed to evaluate the Fourier transforms, but no iterative scheme is required, so the computational burden remains modest.

Regarding uniqueness, the condition reduces precisely to requiring that the symbol of the WH operator have index zero. In Ikeda and Igarashi (2016), the original existence condition due to Blackwell was insufficient to establish uniqueness, which was obtained only after strengthening the condition. In contrast, the present approach offers a more convenient and flexible method. The condition employed here is not only sufficient but also necessary, suggesting that the method may be useful not only for the present model but also for a broader class of models involving integral equations.

4 Conclusion

This paper examined the derivation and uniqueness of solutions to integral equations that frequently arise in finance and macroeconomics, using Fourier-transform techniques within a structural credit-risk framework. The equation arising in the present model is a WH integral equation, one of the most intricate types in this class. We showed that if the symbol of the associated operator has index zero, then an analytic solution exists and is uniquely determined.

Although in the present model the solution cannot be expressed in a fully explicit closed form, the method demonstrates that solutions—previously obtained only through iterative schemes—can instead be computed by numerically evaluating Fourier-transformed expressions, thereby reducing computational burden. Regarding uniqueness, our argument establishes it under a condition weaker than Blackwell's, which highlights the flexibility and practical value of this approach. Future research will investigate how broadly this method can be applied to other models involving integral equations.

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